Lecture 21

Tuesday, February 22, 2022 8:00 PM

* Prayer
* Spiritual thought

Optimization problem:

$$f(x,y,\dots) \longrightarrow \min/\max$$

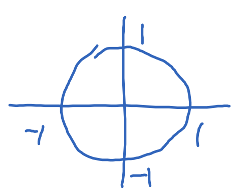
Critical points: $\nabla f = 0$

Local min/max: second derivative test: $\begin{cases} f_{xx}f_{yy} - f_{xy}^2 > 0,\ f_{xx} > 0 : \text{local min} \\ \quad\text{"}\qquad\qquad\quad\ ,\ f_{xx} < 0 : \text{local max} \\ \quad\text{"}\qquad < 0 : \text{saddle point} \end{cases}$

Global min/max doesn't always exist.

If they do, they are either attained at a local extreme or on the boundary.

Ex $f(x,y) = x^3 + y^3$ on $D = \{(x,y) : x^2 + y^2 \le 1\}$

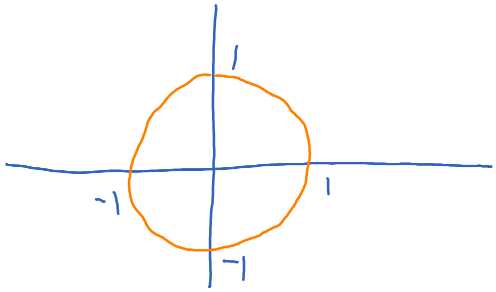


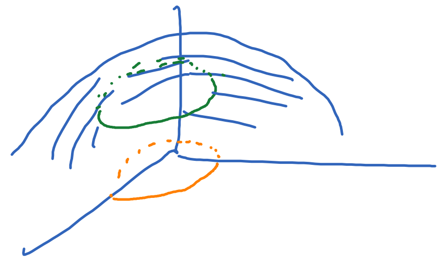
Find min/max of f in D.

Procedure:
- find critical points of f in D
- find min/max of f on the boundary of D
- compare f at those values

Sometimes, it is not easy to extract one variable from the equation of the boundary of D. In this case, $x^2+y^2=1$.

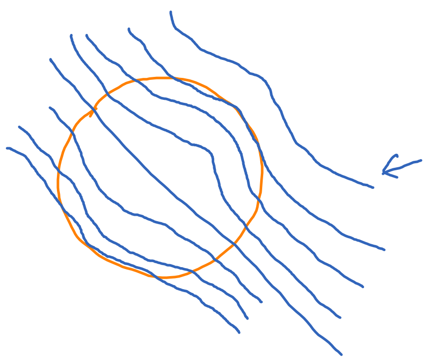
Finding min/max of f on the boundary of D is equivalent to finding min/max of x^3+y^3 under the constraint $x^2+y^2=1$.





we want to find the highest/lowest points on the green curve.

There is a visual way to do this on Mathematica:



level curve of x^3+y^3

We are only interested in the level curves that intersect the circle. We are particularly interested in the level curve that corresponds to the largest value of x^3+y^3.

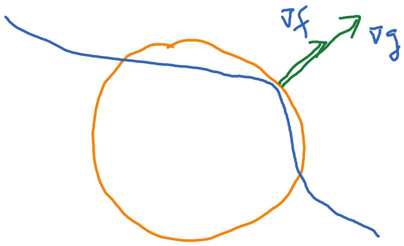
On Mathematica:

```
p1 = ContourPlot[x^3+y^3, {x,-1,1}, {y,-1,1}, Contours -> 100]
p2 = ContourPlot[x^2+y^2 == 1, {x,-1,1}, {y,-1,1}, ContourStyle -> Red]
Show[p1, p2]
```

* Observation: the level curve that touches the circle at only one point is the one corresponds to min/max of x^3+y^3.

* Solve analytically:

$$f(x,y) = x^3+y^3$$
$$g(x,y) = x^2+y^2 = 1$$



Solve for x, y, λ from

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 1 \end{cases}$$

λ is called a Lagrange multiplier

$$\rightsquigarrow \begin{cases} \langle 3x^2, 3y^2 \rangle = \lambda \langle 2x, 2y \rangle \\ x^2+y^2=1 \end{cases}$$

$$\rightsquigarrow \begin{cases} 3x^2 = 2\lambda x \\ 3y^2 = 2\lambda y \\ x^2+y^2=1 \end{cases} \rightsquigarrow \begin{cases} x(3x-2\lambda) = 0 \\ y(3y-2\lambda) = 0 \\ x^2+y^2=1 \end{cases}$$

If $x=0$ then $y=\pm 1$ and $3y-2\lambda=0$.

$$\begin{cases} x=0 \\ y=1 \\ \lambda = 3/2 \end{cases} \quad \text{or} \quad \begin{cases} x=0 \\ y=-1 \\ y=-3/2 \end{cases}$$

If $y=0$ then $x=\pm 1$ and $3x-2\lambda=0$

$$\begin{cases} x=1 \\ y=0 \\ \lambda = \frac{3}{2} \end{cases} \quad \text{or} \quad \begin{cases} x=-1 \\ y=0 \\ \lambda = -3/2 \end{cases}$$

If $x \neq 0, y \neq 0$ then $x = y = \frac{2\lambda}{3}$ and

$$1 = x^2 + y^2 = \frac{4\lambda^2}{9} + \frac{4\lambda^2}{9} = \frac{8\lambda^2}{9}$$

$$\rightsquigarrow \lambda = \pm\sqrt{\frac{9}{8}} = \pm\frac{3\sqrt{2}}{4}$$

$$\rightsquigarrow \begin{cases} x = \frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{2} \end{cases} \text{ or } \begin{cases} x = -\frac{\sqrt{2}}{2} \\ y = -\frac{\sqrt{2}}{2} \end{cases}$$

Compare: values of f at these points

	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$f \downarrow$	1	-1	1	-1	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$

$\min_D f = -1$, attained at $(-1,0), (0,-1)$

$\max_D f = 1$, attained at $(1,0), (0,1)$.

More than one constraint:

Each constraint corresponds to a Lagrange multiplier.

$$\begin{cases} x^3 + y^3 + z^3 \to \text{min/max} \\ x + y + z = 1 \\ x^2 + y^2 + z^2 = 2 \end{cases}$$

$f(x,y,z) = x^3 + y^3 + y^3$

$g(x,y,z) = x + y + z$

$h(x,y,z) = x^2 + y^2 + z^2$

Solve for x, y, z, λ, μ:
$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 1 \\ h = 2 \end{cases}$$